

## SELF SIMILARITY OF GENERALIZED CANTOR SETS

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ABSTRACT. We consider the self-similar structure of the class of generalized Cantor sets

$$\Gamma_{\beta, \mathcal{D}} = \left\{ \sum_{n=1}^{\infty} d_n \beta^n : d_n \in D_n, n \geq 1 \right\},$$

where  $0 < \beta < 1$  and  $D_n, n \geq 1$ , are nonempty and finite subsets of  $\mathbb{Z}$ . We give a characterization for  $\Gamma_{\beta, \mathcal{D}}$  to be a self similar set generated by an iterated function system (IFS)  $\{f_i(x) = rx + a_i : i \in I\}$  with  $0 < |r| < 1$ . Moreover, we show that this characterization is equivalent to  $\Gamma_{\beta, \mathcal{D}}$  being a self-similar set generated by an IFS  $\{g_j(x) = rx + b_j : j \in J\}$  with  $r = \beta^q$  for some  $q \in \mathbb{Z}^+$ . When  $\beta$  is smaller, we show that this characterization is also equivalent to  $\Gamma_{\beta, \mathcal{D}}$  being a self-similar set generated by an IFS  $\{h_k(x) = r_k x + c_k : k \in K\}$  with  $r_k = \beta^{q_k}$  for some  $q_k \in \mathbb{Z}^+$ . An application to the self-similarity of intersections of generalized Cantor sets will be given.

*Key words.* generalized Cantor sets, homogeneously generated self-similar sets, strongly eventually periodic, iterated function systems, intersections of Cantor sets.

**MSC:** 28A80, 28A78, 37B10

## 1. INTRODUCTION

For an integer  $d$ , let  $\phi_d$  be a contractive map defined by

$$(1) \quad \phi_d(x) = \beta(x + d), \quad x \in \mathbb{R},$$

where  $0 < \beta < 1$ . Let  $\mathbb{Z}^\infty$  be the set of infinite sequences  $\{d_n\}_{n=1}^\infty$  with each  $d_n \in \mathbb{Z}$ . We define the coding map  $\pi : \mathbb{Z}^\infty \rightarrow \mathbb{R}$  by

$$(2) \quad \pi(\{d_n\}_{n=1}^\infty) := \lim_{m \rightarrow \infty} \phi_{d_1} \circ \cdots \circ \phi_{d_m}(0) = \sum_{n=1}^{\infty} d_n \beta^n.$$

For  $n \geq 1$ , let  $D_n$  be a nonempty and finite subset of  $\mathbb{Z}$ , and let  $\mathcal{D} = \bigotimes_{n=1}^{\infty} D_n$  be the set of infinite sequences  $\{d_n\}_{n=1}^\infty$  with  $d_n \in D_n$  for all  $n \geq 1$ . Then the *generalized Cantor set*  $\Gamma_{\beta, \mathcal{D}}$  of type  $\mathcal{D}$  is defined as the image set of  $\mathcal{D}$  under the coding map  $\pi$ , i.e.,

$$(3) \quad \Gamma_{\beta, \mathcal{D}} := \pi(\mathcal{D}) = \left\{ \sum_{n=1}^{\infty} d_n \beta^n : d_n \in D_n, n \geq 1 \right\}.$$

We assume that the digit sets  $D_n, n \geq 1$ , have bounded cardinality and the sums in (3) are all convergent, which we express as

$$(4) \quad \sup_{n \geq 1} \max_{d, d' \in D_n} (d - d') < \infty \quad \text{and} \quad \left| \sum_{i=1}^{\infty} \lambda_n \beta^n \right| < \infty,$$

where  $\lambda_n = \max\{d : d \in D_n\}$ .

The generalized Cantor set  $\Gamma_{\beta, \mathcal{D}}$  of type  $\mathcal{D} = \bigotimes_{n=1}^{\infty} D_n$  can also be looked at in a geometrical way. To illustrate this geometrical construction, we need the extra assumptions:

$$\gamma_{\mathcal{D}} := \inf\{\gamma_n : n \geq 1\} > -\infty, \quad \lambda_{\mathcal{D}} := \sup\{\lambda_n : n \geq 1\} < \infty,$$

where  $\gamma_n = \min\{d : d \in D_n\}$ . Then all the digit sets  $D_n \subseteq \{\gamma_{\mathcal{D}}, \gamma_{\mathcal{D}} + 1, \dots, \lambda_{\mathcal{D}}\}, n \geq 1$ . Let  $F_0 = [\gamma_{\mathcal{D}}\beta/(1-\beta), \lambda_{\mathcal{D}}\beta/(1-\beta)]$  and then, for  $n \geq 1$ , inductively define

$$F_n = \bigcup_{d \in D_n} \phi_d(F_{n-1}),$$

where  $\phi_d$  is the contractive map defined in (1). Clearly,  $\{F_n\}_{n=0}^{\infty}$  is a monotonic decreasing sequence of compact subsets of  $\mathbb{R}$ . Then the generalized Cantor set  $\Gamma_{\beta, \mathcal{D}}$  can be written as

$$\Gamma_{\beta, \mathcal{D}} = \bigcap_{n=0}^{\infty} F_n.$$

For example, let  $\beta = 1/4, D_1 = \{0, 2\}, D_2 = \{1, 2\}$  and  $D_n = \{0, 1, 2\}$  for all  $n \geq 3$ . Then  $\gamma_{\mathcal{D}} = 0, \lambda_{\mathcal{D}} = 2$ , and the first few generations  $F_0, \dots, F_3$  of  $\Gamma_{1/4, \mathcal{D}}$  are plotted in Figure 1.

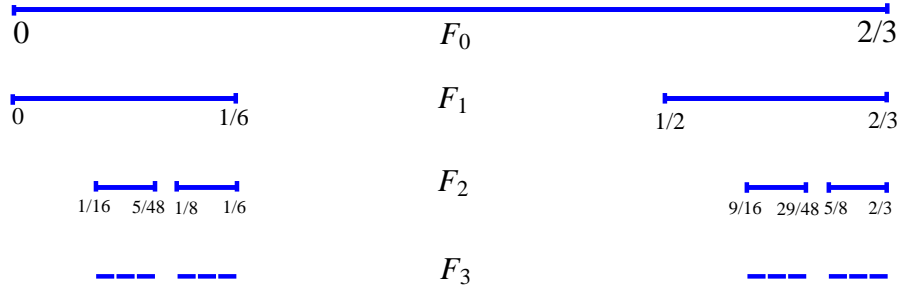


FIGURE 1. The first few generations  $F_0, F_1, F_2, F_3$  of the generalized Cantor set  $\Gamma_{\beta, \mathcal{D}}$  of type  $\mathcal{D} = \bigotimes_{n=1}^{\infty} D_n$  with  $\beta = 1/4, D_1 = \{0, 2\}, D_2 = \{1, 2\}$  and  $D_n = \{0, 1, 2\}$  for all  $n \geq 3$ .

A map  $S : \mathbb{R} \rightarrow \mathbb{R}$  is called a *similitude* if there is  $r, 0 < |r| < 1$ , and  $b \in \mathbb{R}$  such that  $S(x) = rx + b$  for  $x \in \mathbb{R}$ . Here  $r$  is called the *contraction ratio* of  $S$ . Suppose  $\{S_j(x) = r_j x + b_j : j \in J\}$ , where  $J$  is a finite index set. Then there exists a compact set  $\Lambda$  satisfying

$$\Lambda = \bigcup_{j \in J} S_j(\Lambda).$$

The compact set  $\Lambda$  is called a *self-similar set* generated by the *iterated function system* (IFS)  $\{S_j : j \in J\}$ .

The generalized Cantor set  $\Gamma_{\beta, \mathcal{D}}$  of type  $\mathcal{D} = \bigotimes_{n=1}^{\infty} D_n$  reduces to a self-similar set when all the digit sets  $D_n = D, n \geq 1$ , for some finite set  $\emptyset \neq D \subseteq \mathbb{Z}$ , i.e.,  $\Gamma_{\beta, \mathcal{D}} = \bigcup_{d \in D} \phi_d(\Gamma_{\beta, \mathcal{D}})$ . When all the digit sets  $D_n, n \geq 1$ , are *consecutive*, i.e., there exists some  $\tau_n \geq 0$  such that

$$D_n = \{\gamma_n, \gamma_n + 1, \dots, \gamma_n + \tau_n\} \subseteq \mathbb{Z},$$

the authors in [KLD10] and [LYZ11] gave a necessary and sufficient condition for which  $\Gamma_{\beta, \mathcal{D}}$  is a self-similar set — in fact we have more equivalent conditions as described in Theorem 2.1. However, when some digit set  $D_n$  is not consecutive, nothing is known about the self-similar structure of  $\Gamma_{\beta, \mathcal{D}}$ . It is natural to ask: to what extent can a generalized Cantor set  $\Gamma_{\beta, \mathcal{D}}$  of type  $\mathcal{D}$  still be a self-similar set?

Another motivation to study the self-similar structure of generalized Cantor sets dates back to the study of intersections of Cantor set and its translations, which plays an important role in planar homoclinic bifurcations from nonlinear dynamical systems (cf. [PT93], see also [DH95]).

We arrange the paper in the following way. In Section 2 we state the main results. The proof of these results are given in Section 3, 4, 5. In Section 6 we consider an application to the self-similarity of intersections of generalized Cantor sets, and we give some final remarks and open questions in Section 7.

## 2. PRELIMINARIES AND THE MAIN RESULTS

For  $0 < \beta < 1$ , let  $\Gamma_{\beta, \mathcal{D}}$  be a generalized Cantor set of type  $\mathcal{D} = \bigotimes_{n=1}^{\infty} D_n$ . Recall from Equation (2) and (3) that  $\pi$  is the surjective map from  $\mathcal{D}$  to  $\Gamma_{\beta, \mathcal{D}}$  defined by

$$\pi(\{d_n\}_{n=1}^{\infty}) = \sum_{i=1}^{\infty} d_n \beta^n.$$

The infinite sequence  $\{d_n\}_{n=1}^{\infty} \in \mathcal{D}$  is called a  $\mathcal{D}$ -code of  $\pi(\{d_n\}_{n=1}^{\infty}) \in \Gamma_{\beta, \mathcal{D}}$ . We point out that a point  $x \in \Gamma_{\beta, \mathcal{D}}$  may have multiple  $\mathcal{D}$ -codes. But when  $0 < \beta < 1/N_{\mathcal{D}}$ , the map  $\pi$  from  $\mathcal{D}$  to  $\Gamma_{\beta, \mathcal{D}}$  is bijective and, hence, each point in  $\Gamma_{\beta, \mathcal{D}}$  has a unique  $\mathcal{D}$ -code. Here  $N_{\mathcal{D}}$  is the *span* of  $\mathcal{D}$  defined as

$$(5) \quad N_{\mathcal{D}} := \sup_{n \geq 1} \max_{d, d' \in D_n} (1 + d - d').$$

Obviously,  $N_{\mathcal{D}} \geq 1$ . If  $N_{\mathcal{D}} = 1$ , then  $\Gamma_{\beta, \mathcal{D}}$  contains only a single point. Excluding this trivial case and using the assumption in (4), we have  $2 \leq N_{\mathcal{D}} < \infty$ . If no confusion arises about  $\mathcal{D}$ , we will write  $N$  instead of  $N_{\mathcal{D}}$ . Let  $\Omega_N := \{0, 1, \dots, N-1\}$ .

**Definition 2.1** (Deng, He and Wen [DHW08]). *A sequence  $\{d_n\}_{n=1}^{\infty} \in \Omega_N^{\infty}$  is called strongly eventually periodic (or simply, SEP) with period  $p \in \mathbb{Z}^+$  if there exist two finite sequences  $\{a_{\ell}\}_{\ell=1}^p, \{b_{\ell}\}_{\ell=1}^p \in \Omega_N^p$  such that*

$$\{d_n\}_{n=1}^{\infty} = \{a_{\ell}\}_{\ell=1}^p \overline{\{a_{\ell} + b_{\ell}\}_{\ell=1}^p},$$

where  $\overline{\{c_{\ell}\}_{\ell=1}^p} \in \Omega_N^{\infty}$  stands for the infinite repetition of a finite sequence  $\{c_{\ell}\}_{\ell=1}^p \in \Omega_N^p$ .

Obviously, a periodic sequence  $\{d_n\}_{n=1}^\infty$  is a SEP sequence, and a SEP sequence  $\{d_n\}_{n=1}^\infty$  is eventually periodic. Note that a SEP sequence  $\{d_n\}_{n=1}^\infty$  is called strongly periodic in [DHW08].

Analogously, we have the definition for the strong eventual periodicity of a sequence of sets.

**Definition 2.2.** A sequence of sets  $\{D_n\}_{n=1}^\infty$  with  $\emptyset \neq D_n \subseteq \mathbb{Z}$  is called strongly eventually periodic (or simply, SEP) with period  $p \in \mathbb{Z}^+$  if there exist two finite sequences of sets  $\{A_\ell\}_{\ell=1}^p, \{B_\ell\}_{\ell=1}^p$  such that

$$\{D_n\}_{n=1}^\infty = \{A_\ell\}_{\ell=1}^p \overline{\{A_\ell + B_\ell\}_{\ell=1}^p},$$

where  $A + B = \{a + b : a \in A, b \in B\}$  and  $\overline{\{C_\ell\}_{\ell=1}^p}$  denotes the infinite repetition of a finite sequence of sets  $\{C_\ell\}_{\ell=1}^p$ .

When a sequence of sets  $\{D_n\}_{n=1}^\infty$  is SEP with period  $p$ , it is easy to check that the sequence  $\{|D_n| - 1\}_{n=1}^\infty$  is also SEP with period  $p$ , where  $|A|$  stands for the cardinality of a set  $A$ . But this is not true the other way around.

When all the digit sets  $D_n, n \geq 1$ , are consecutive, the authors in [KLD10] and [LYZ11] showed that the self-similar structure of  $\Gamma_{\beta, \mathcal{D}}$  can be characterized by the SEP of the sequence  $\{|D_n| - 1\}_{n=1}^\infty$ . In the following theorem we will show that this can also be characterized by the SEP of the sequence of sets  $\{D_n - \gamma_n\}_{n=1}^\infty$ , where  $\gamma_n = \min\{d : d \in D_n\}$  and  $A - b := A + \{-b\} = \{a - b : a \in A\}$  for a set  $A$  and a real number  $b$ .

**Theorem 2.1.** Let  $\Gamma_{\beta, \mathcal{D}}$  be a generalized Cantor set of type  $\mathcal{D} = \bigotimes_{n=1}^\infty D_n$  in (3) with span  $N = N_{\mathcal{D}} \geq 2$ . Suppose that all the digit sets  $D_n, n \geq 1$ , are consecutive, and  $0 < \beta < 1/N$ . Then the following conditions are equivalent.

- (i)  $\Gamma_{\beta, \mathcal{D}}$  is a self-similar set;
- (ii)  $\Gamma_{\beta, \mathcal{D}}$  is a self-similar set generated by an IFS  $\{f_i(x) = rx + a_i : i \in I\}$ ;
- (iii)  $\Gamma_{\beta, \mathcal{D}}$  is a self-similar set generated by an IFS  $\{g_j(x) = rx + b_j : j \in J\}$  with  $r = \beta^q$  for some  $q \in \mathbb{Z}^+$ ;
- (iv) the sequence  $\{|D_n| - 1\}_{n=1}^\infty$  is SEP;
- (v) the sequence of sets  $\{D_n - \gamma_n\}_{n=1}^\infty$  is SEP, where  $\gamma_n = \min\{d : d \in D_n\}$ .

When some digit set  $D_n$  is not consecutive, we will show in Theorem 2.2 that the self-similar structure of  $\Gamma_{\beta, \mathcal{D}}$  still can be characterized by the SEP of the sequence of sets  $\{D_n - \gamma_n\}_{n=1}^\infty$ . Since for  $N = 2$  that all the digit sets  $D_n, n \geq 1$ , are consecutive, we will assume  $N \geq 3$  in the following theorem.

**Theorem 2.2.** Let  $\Gamma_{\beta, \mathcal{D}}$  be a generalized Cantor set of type  $\mathcal{D} = \bigotimes_{n=1}^\infty D_n$  in (3) with span  $N = N_{\mathcal{D}} \geq 3$ . Suppose  $0 < \beta \leq 1/[(3N - 1)/2]$ , where  $[x]$  denotes the integer part of a real number  $x$ . Then the following conditions are equivalent.

- (I)  $\Gamma_{\beta, \mathcal{D}}$  is a self-similar set generated by an IFS  $\{f_i(x) = rx + a_i : i \in I\}$ ;
- (II)  $\Gamma_{\beta, \mathcal{D}}$  is a self-similar set generated by an IFS  $\{g_j(x) = rx + b_j : j \in J\}$  with  $r = \beta^q$  for some  $q \in \mathbb{Z}^+$ ;

(III) the sequence of sets  $\{D_n - \gamma_n\}_{n=1}^\infty$  is SEP, where  $\gamma_n = \min\{d : d \in D_n\}$ .

We point out that the upper bound  $1/[(3N-1)/2]$  for  $\beta$  in Theorem 2.2 can not be improved to  $1/N$  as in Theorem 2.1 (see Example 7.1).

Moreover, when  $0 < \beta \leq 1/(2N-1)$ , we will show in Theorem 2.3 that the SEP of  $\{D_n - \gamma_n\}_{n=1}^\infty$  is also an characterization for  $\Gamma_{\beta, \mathcal{D}}$  to be a self-similar set generated by an IFS  $\{h_k(x) = r_k x + c_k : k \in K\}$  with  $r_k = \beta^{q_k}$  for some  $q_k \in \mathbb{Z}^+$ .

**Theorem 2.3.** *Let  $\Gamma_{\beta, \mathcal{D}}$  be a generalized Cantor set of type  $\mathcal{D} = \bigotimes_{n=1}^\infty D_n$  in (3) with span  $N = N_{\mathcal{D}} \geq 2$ . Suppose  $0 < \beta \leq 1/(2N-1)$ . Then the following conditions are equivalent.*

- (A)  $\Gamma_{\beta, \mathcal{D}}$  is a self-similar set generated by an IFS  $\{f_i(x) = rx + a_i : i \in I\}$ ;
- (B)  $\Gamma_{\beta, \mathcal{D}}$  is a self-similar set generated by an IFS  $\{g_j(x) = rx + b_j : j \in J\}$  with  $r = \beta^q$  for some  $q \in \mathbb{Z}^+$ ;
- (C)  $\Gamma_{\beta, \mathcal{D}}$  is a self-similar set generated by an IFS  $\{h_k(x) = r_k x + c_k : k \in K\}$  with  $r_k = \beta^{q_k}$  for some  $q_k \in \mathbb{Z}^+$ ;
- (D) the sequence of sets  $\{D_n - \gamma_n\}_{n=1}^\infty$  is SEP, where  $\gamma_n = \min\{d : d \in D_n\}$ .

We remark here that Pedersen and Phillips prove (independently of us) in [PP12, Theorem 1.2] that (B)  $\Leftrightarrow$  (D). However, our results in Theorem 2.3 is more general. Moreover, Theorem 2.2 generalizes [PP12, Theorem 1.2] to a larger class of  $\beta$ , i.e.,  $0 < \beta < 1/[(3N-1)/2]$ .

To prove Theorem 2.1 — 2.3, it is convenient to shift  $\Gamma_{\beta, \mathcal{D}}$  such that 0 is the left endpoint. More explicitly, let

$$D'_n := D_n - \gamma_n \quad \text{with} \quad \gamma_n = \min\{d : d \in D_n\}.$$

Then  $0 \in D'_n \subseteq \Omega_N$  for all  $n \geq 1$ . Accordingly, let

$$\mathcal{D}' := \bigotimes_{n=1}^\infty D'_n = \bigotimes_{n=1}^\infty (D_n - \gamma_n).$$

Then the generalized Cantor set  $\Gamma_{\beta, \mathcal{D}'}$  of type  $\mathcal{D}' = \bigotimes_{n=1}^\infty D'_n$  is a translation of  $\Gamma_{\beta, \mathcal{D}}$ , since by using Equation (2) and (3) we have

$$\begin{aligned} \Gamma_{\beta, \mathcal{D}'} &= \pi(\mathcal{D}') = \left\{ \sum_{n=1}^\infty d_n \beta^n : d_n \in D_n - \gamma_n \right\} \\ (6) \quad &= \left\{ \sum_{n=1}^\infty (d_n - \gamma_n) \beta^n : d_n \in D_n \right\} = \Gamma_{\beta, \mathcal{D}} - \pi(\{\gamma_n\}_{n=1}^\infty). \end{aligned}$$

So it suffices to prove Theorem 2.1 — 2.3 for  $\Gamma_{\beta, \mathcal{D}'}$  instead of  $\Gamma_{\beta, \mathcal{D}}$ . The following property makes it more convenient to deal with  $\Gamma_{\beta, \mathcal{D}'}$ .

**(P1)** Since  $0 \in D'_n$  for all  $n \geq 1$ , we have  $\sum_{n=1}^m d_n \beta^n = \pi(d_1 \cdots d_m \bar{0}) \in \Gamma_{\beta, \mathcal{D}'}$  for any  $m \geq 1$  and for any  $d_\ell \in D'_\ell$ ,  $\ell = 1, \dots, m$ . In particular,  $0 = \pi(\bar{0}) \in \Gamma_{\beta, \mathcal{D}'}$ .

## 3. PROOF OF THEOREM 2.1

The proof of Theorem 2.1 is based on the following lemma. Inspired by the proof of [LYZ11, Theorem 1.2], we first show (v)  $\Rightarrow$  (iii) for more general digit sets  $D'_n, n \geq 1$  and a larger class of  $\beta$ 's.

**Lemma 3.1.** *Let  $\Gamma_{\beta, \mathcal{D}'}$  be a generalized Cantor set of type  $\mathcal{D}' = \bigotimes_{n=1}^{\infty} D'_n$  as in (6). Suppose  $0 < \beta < 1$ . If the sequence of sets  $\{D'_n\}_{n=1}^{\infty}$  is SEP, then  $\Gamma_{\beta, \mathcal{D}'}$  is a self-similar set generated by an IFS  $\{g_j(x) = rx + b_j : j \in J\}$  with  $r = \beta^q$  for some  $q \in \mathbb{Z}^+$ .*

*Proof.* By Definition 2.2, we assume  $\bigotimes_{n=1}^{\infty} D'_n = \left( \bigotimes_{\ell=1}^q A_{\ell} \right) \otimes \left( \bigotimes_{\ell=1}^q (A_{\ell} + B_{\ell}) \right)^{\infty}$  for some  $q \in \mathbb{Z}^+$ , where  $\left( \bigotimes_{\ell=1}^q C_{\ell} \right)^{\infty} := \bigotimes_{k=1}^{\infty} \left( \bigotimes_{\ell=1}^q C_{\ell} \right)$ . The proposition will then follow by showing that  $\Gamma_{\beta, \mathcal{D}'}$  is a self-similar set generated by the IFS  $\{g_e(x) = \beta^q x + e : e \in \mathcal{E}\}$ , where

$$\mathcal{E} = \left\{ \sum_{\ell=1}^{2q} d_{\ell} \beta^{\ell} : \{d_{\ell}\}_{\ell=1}^{2q} \in \left( \bigotimes_{\ell=1}^q A_{\ell} \right) \otimes \left( \bigotimes_{\ell=1}^q B_{\ell} \right) \right\}.$$

Let  $\Lambda$  be the self-similar set generated by the IFS  $\{g_e : e \in \mathcal{E}\}$ . Then it is easy to write  $\Lambda$  in an algebraic way as

$$\begin{aligned} \Lambda &= \left\{ \lim_{m \rightarrow \infty} g_{e_1} \circ \cdots \circ g_{e_m}(0) : e_n \in \mathcal{E}, n \geq 1 \right\} \\ (7) \quad &= \left\{ \sum_{n=1}^{\infty} e_n \beta^{q(n-1)} : e_n \in \mathcal{E}, n \geq 1 \right\}. \end{aligned}$$

Let  $x \in \Gamma_{\beta, \mathcal{D}'}$ , and let  $\{x_n\}_{n=1}^{\infty}$  be a  $\mathcal{D}'$ -code of  $x$  (here  $x$  may have multiple  $\mathcal{D}'$ -codes when  $\beta \geq 1/N$ ), i.e.,  $\pi(\{x_n\}_{n=1}^{\infty}) = x$  and

$$\{x_n\}_{n=1}^{\infty} \in \mathcal{D}' = \bigotimes_{n=1}^{\infty} D'_n = \left( \bigotimes_{\ell=1}^q A_{\ell} \right) \otimes \left( \bigotimes_{\ell=1}^q (A_{\ell} + B_{\ell}) \right)^{\infty},$$

where  $\pi$  is the coding map defined in (2). So  $\{x_{\ell}\}_{\ell=1}^q \in \bigotimes_{\ell=1}^q A_{\ell}$ , and for any  $m \geq 1$ ,  $\{x_{mq+\ell}\}_{\ell=1}^q$  can be represented as

$$(8) \quad \{x_{mq+\ell}\}_{\ell=1}^q = \{a_{mq+\ell} + b_{mq+\ell}\}_{\ell=1}^q,$$

where  $a_{mq+\ell} \in A_{\ell}, b_{mq+\ell} \in B_{\ell}$  for  $1 \leq \ell \leq q$ . Hence, by using Equation (2), (8) and (7) successively, we have

$$\begin{aligned} x &= \pi(\{x_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} x_n \beta^n = \sum_{\ell=1}^q x_{\ell} \beta^{\ell} + \sum_{m=1}^{\infty} \sum_{\ell=1}^q (a_{mq+\ell} + b_{mq+\ell}) \beta^{mq+\ell} \\ &= \left( \sum_{\ell=1}^q x_{\ell} \beta^{\ell} + \sum_{\ell=q+1}^{2q} b_{\ell} \beta^{\ell} \right) + \sum_{m=1}^{\infty} \beta^{mq} \left( \sum_{\ell=1}^q a_{mq+\ell} \beta^{\ell} + \sum_{\ell=q+1}^{2q} b_{mq+\ell} \beta^{\ell} \right) \\ &\in \Lambda. \end{aligned}$$

This implies  $\Gamma_{\beta, \mathcal{D}'} \subseteq \Lambda$ . The converse inclusion can be shown by going upwards in the last calculation.  $\square$

*Proof of Theorem 2.1.* From [LYZ11, Theorem 1.2] and [KLD10, Theorem 3.2] it follows that (i)  $\Leftrightarrow$  (iv). Obviously, (iii)  $\Rightarrow$  (ii)  $\Rightarrow$  (i). Moreover, Lemma 3.1 yields (v)  $\Rightarrow$  (iii). So, to finish the proof of Theorem 2.1 it suffices to show (iv)  $\Rightarrow$  (v).

We assume for  $n \geq 1$  that  $D'_n = \{0, 1, \dots, \tau_n\} \subseteq \Omega_N$  for some  $\tau_n \geq 0$ . Suppose  $\{|D'_n| - 1\}_{n=1}^\infty = \{\tau_n\}_{n=1}^\infty$  is SEP. Then by Definition 2.1 there exist  $\{a_\ell\}_{\ell=1}^p, \{b_\ell\}_{\ell=1}^p \in \Omega_N^p$  for some  $p \in \mathbb{Z}^+$  such that

$$\{\tau_n\}_{n=1}^\infty = \{a_\ell\}_{\ell=1}^p \overline{\{a_\ell + b_\ell\}_{\ell=1}^p}.$$

This implies that the sequence of sets  $\{D'_n\}_{n=1}^\infty$  is SEP, since

$$(9) \quad \{D'_n\}_{n=1}^\infty = \{\Omega_{\tau_n+1}\}_{n=1}^\infty = \{\Omega_{a_\ell+1}\}_{\ell=1}^p \overline{\{\Omega_{a_\ell+1} + \Omega_{b_\ell+1}\}_{\ell=1}^p},$$

where  $\Omega_s = \{0, 1, \dots, s-1\}$ . □

#### 4. PROOF OF THEOREM 2.2

Obviously, (II)  $\Rightarrow$  (I). We will prove (I)  $\Rightarrow$  (II) in the following lemma by using the same idea as in the proof of [KLD10, Theorem 3.2]. Here we point out that the digit sets  $D_n, n \geq 1$ , do not have to be consecutive.

**Lemma 4.1.** *Let  $\Gamma_{\beta, \mathcal{D}'}$  be a generalized Cantor set of type  $\mathcal{D}' = \bigotimes_{n=1}^\infty D'_n$  as in (6) with span  $N = N_{\mathcal{D}'} \geq 2$ . Suppose  $0 < \beta < 1/N$ . If  $\Gamma_{\beta, \mathcal{D}'}$  is a self-similar set generated by an IFS  $\{f_i(x) = rx + a_i : i \in I\}$ , then  $\Gamma_{\beta, \mathcal{D}'}$  can also be generated by an IFS  $\{g_j(x) = rx + b_j : j \in J\}$  with  $r = \beta^q$  for some  $q \in \mathbb{Z}^+$ .*

*Proof.* Suppose  $\Gamma_{\beta, \mathcal{D}'}$  is a self-similar set generated by an IFS  $\{f_i(x) = rx + a_i : i \in I\}$  with  $0 < |r| < 1$ , i.e.,

$$\Gamma_{\beta, \mathcal{D}'} = \bigcup_{i \in I} f_i(\Gamma_{\beta, \mathcal{D}'}).$$

One can assume that  $0 < r < 1$ , since otherwise we can consider the IFS  $\{f_i \circ f_{i'}(x) : i, i' \in I\}$  instead of  $\{f_i(x) : i \in I\}$ . Since  $0 < \beta < 1$ , there exists some  $\alpha > 0$  such that  $r = \beta^\alpha$ . If  $\alpha$  is rational, then we take  $m \in \mathbb{Z}^+$  such that  $m\alpha \in \mathbb{Z}^+$ . Thus the lemma follows by taking  $\{g_j : j \in J\} = \{f_{i_1} \circ \dots \circ f_{i_m}(x) : i_n \in I, n = 1, \dots, m\}$ .

We will finish the proof by showing that irrational  $\alpha$  will lead to a contradiction. Take  $m \in \mathbb{Z}^+$  such that

$$(10) \quad 0 < 1 - \beta^{m\alpha - [m\alpha]} < \frac{1 - N\beta}{N(1 - \beta)}.$$

This is possible since  $0 < \beta < 1/N$  and  $\{m\alpha - [m\alpha] : m \in \mathbb{Z}^+\}$  is dense in  $(0, 1)$ . Recall from (P1) that  $0 = \pi(\bar{0}) \in \Gamma_{\beta, \mathcal{D}'}$ . This implies that there exists  $i_0 \in I$  such that  $f_{i_0}(x) = rx$ . Take  $\ell$  such that  $|D'_\ell| > 1$ , and take  $d > 0$  from  $D'_\ell$ . Then by (P1)  $d\beta^\ell = \pi(0^{\ell-1}d\bar{0}) \in \Gamma_{\beta, \mathcal{D}'}$ , where  $0^{\ell-1}$  denotes  $\ell - 1$  times repetition of the digit 0. So

$$(11) \quad f_{i_0}^m(d\beta^\ell) = r^m d\beta^\ell = d\beta^{m\alpha + \ell} < d\beta^{[m\alpha] + \ell} =: \bar{\eta}.$$

Using Equation (10) and  $d < N$ , it follows that

$$\begin{aligned}
f_{i_0}^m(d\beta^\ell) &= d\beta^{m\alpha+\ell} = d\beta^{[m\alpha]+\ell} - d(1 - \beta^{m\alpha-[m\alpha]})\beta^{[m\alpha]+\ell} \\
&> d\beta^{[m\alpha]+\ell} - \frac{d(1 - N\beta)}{N(1 - \beta)}\beta^{[m\alpha]+\ell} > d\beta^{[m\alpha]+\ell} - \frac{1 - N\beta}{1 - \beta}\beta^{[m\alpha]+\ell} \\
&= (d - 1)\beta^{[m\alpha]+\ell} + \frac{(N - 1)\beta^{[m\alpha]+\ell+1}}{1 - \beta} \\
&= (d - 1)\beta^{[m\alpha]+\ell} + \sum_{n=[m\alpha]+\ell+1}^{\infty} (N - 1)\beta^n =: \underline{\eta}.
\end{aligned}$$

This, together with Equation (11), yields that

$$(12) \quad f_{i_0}^m(d\beta^\ell) \in (\underline{\eta}, \bar{\eta}) \cap f_{i_0}^m(\Gamma_{\beta, \mathcal{D}'}) \subseteq (\underline{\eta}, \bar{\eta}) \cap \Gamma_{\beta, \mathcal{D}'}.$$

On the other hand, since  $0 < \beta < 1/N$ , the coding map  $\pi$  is bijective on  $\Omega_N^\infty$ . Then

$$\underline{\eta} = \pi(0^{[m\alpha]+\ell-1}d\bar{0}), \quad \bar{\eta} = \pi(0^{[m\alpha]+\ell-1}(d-1)\overline{(N-1)}),$$

which imply  $(\underline{\eta}, \bar{\eta}) \cap \pi(\Omega_N^\infty) = \emptyset$ . This, together with  $\Gamma_{\beta, \mathcal{D}'} = \pi(\mathcal{D}') \subseteq \pi(\Omega_N^\infty)$ , yields that  $(\underline{\eta}, \bar{\eta}) \cap \Gamma_{\beta, \mathcal{D}'} = \emptyset$ , leading to a contradiction with (12).  $\square$

Lemma 3.1 yields (III)  $\Rightarrow$  (II), and Lemma 4.1 yields (I)  $\Leftrightarrow$  (II). So to finish the proof of Theorem 2.2 it suffices to show (II)  $\Rightarrow$  (III). In this direction, we first prove the following lemma.

**Lemma 4.2.** *Let  $\Gamma_{\beta, \mathcal{D}'}$  be a generalized Cantor set of type  $\mathcal{D}' = \bigotimes_{n=1}^\infty D'_n$  as in (6) with span  $N = N_{\mathcal{D}'} \geq 2$ . Suppose  $0 < \beta < 1/N$ . If  $\Gamma_{\beta, \mathcal{D}'}$  is a self-similar set generated by an IFS  $\{g_j(x) = rx + b_j : j \in J\}$  with  $r = \beta^q$  for some  $q \in \mathbb{Z}^+$ , then the sequence of sets  $\{D'_n\}_{n=1}^\infty$  satisfies  $\{D'_n\}_{n=1}^\infty = \{D'_\ell\}_{\ell=1}^p \overline{\{D'_{p+\ell}\}_{\ell=1}^p}$  for some  $p \in \mathbb{Z}^+$ . Moreover,  $D'_\ell \subseteq D'_{p+\ell}$  for  $\ell = 1, \dots, p$ .*

*Proof.* Recall from (P1) that  $0 \in \Gamma_{\beta, \mathcal{D}'}$ . Then there exists  $j_0 \in J$  such that  $g_{j_0}(x) = \beta^q x$ . For  $\ell \geq 1$ , let  $d \in D'_\ell$ , and then by (P1) we take  $d\beta^\ell = \pi(0^{\ell-1}d\bar{0}) \in \Gamma_{\beta, \mathcal{D}'}$ . Then

$$g_{j_0}(d\beta^\ell) = d\beta^{q+\ell} \in \Gamma_{\beta, \mathcal{D}'}.$$

This implies  $d \in D'_{q+\ell}$ , since, by  $0 < \beta < 1/N$ , any point in  $\Gamma_{\beta, \mathcal{D}'}$  has a unique  $\mathcal{D}'$ -code. So  $D'_\ell \subseteq D'_{q+\ell}$ . By iteration, this yields

$$D'_\ell \subseteq D'_{q+\ell} \subseteq \dots \subseteq D'_{mq+\ell} \subseteq \dots$$

for all  $1 \leq \ell \leq q$  and  $m \geq 1$ .

Since  $D'_n \subseteq \Omega_N$  for all  $n \geq 1$ , there exists some large  $m_* \geq 1$  which can be chosen independent of  $\ell$ , such that  $D'_{m_*q+\ell} = D'_{mq+\ell}$  for all  $1 \leq \ell \leq q$  and  $m \geq m_*$ . Take  $p = m_*q$ . Then

$$\{D'_n\}_{n=1}^\infty = \{D'_\ell\}_{\ell=1}^p \overline{\{D'_{p+\ell}\}_{\ell=1}^p}.$$

Clearly,  $D'_\ell \subseteq D'_{p+\ell}$  for  $\ell = 1, \dots, p$ . This completes the proof.  $\square$



Recall from Section 2 that an infinite sequence  $\{x_n\}_{n=1}^\infty \in \mathcal{D}'$  is called a  $\mathcal{D}'$ -code of  $x \in \Gamma_{\beta, \mathcal{D}'}$  if  $x = \sum_{n=1}^\infty x_n \beta^n$  with  $x_n \in D'_n$  for all  $n \geq 1$ . Since  $0 < \beta < 1/N$ , we have  $N \leq [1/\beta]$ . Then  $\mathcal{D}' \subseteq \Omega_N^\infty \subseteq \Omega_{[1/\beta]}^\infty$ , and so  $\Gamma_{\beta, \mathcal{D}'} \subseteq \Gamma_{\beta, \Omega_{[1/\beta]}^\infty}$ . In this case,  $\{x_n\}_{n=1}^\infty$  is also called a  $\Omega_{[1/\beta]}^\infty$ -code of  $x$ .

**Lemma 4.3.** *Let  $\Gamma_{\beta, \mathcal{D}'}$  be a generalized Cantor set of type  $\mathcal{D}' = \bigotimes_{n=1}^\infty D'_n$  in (6) with span  $N = N_{\mathcal{D}'} \geq 2$ . Suppose  $0 < \beta < 1/N$ . If  $x \in \Gamma_{\beta, \mathcal{D}'} \subseteq \Gamma_{\beta, \Omega_{[1/\beta]}^\infty}$  has a  $\Omega_{[1/\beta]}^\infty$ -code  $\{x_n\}_{n=1}^\infty$  which is not of the form  $d_1 \cdots d_k \overline{([1/\beta] - 1)}$ , then  $\{x_n\}_{n=1}^\infty$  is the unique  $\mathcal{D}'$ -code of  $x$ .*

*Proof.* The lemma follows by the fact that when  $1/\beta \notin \mathbb{Z}^+$  the coding map  $\pi$  from  $\Omega_{[1/\beta]}^\infty$  to  $\Gamma_{\beta, \Omega_{[1/\beta]}^\infty}$  is bijective, and when  $1/\beta \in \mathbb{Z}^+$  then  $\pi$  is almost bijective in the sense that only countably many points in  $\Gamma_{\beta, \Omega_{[1/\beta]}^\infty}$  have two  $\Omega_{[1/\beta]}^\infty$ -codes of the forms  $d_1 \cdots d_{k-1} d_k \overline{([1/\beta] - 1)}$  and  $d_1 \cdots d_{k-1} (d_k + 1) \bar{0}$  for some  $d_k + 1 \leq [1/\beta] - 1$ .  $\square$

It is convenient to introduce the following notations. For a sequence of sets  $\{A_n\}_{n=1}^\infty$ , let

$$\sum_{n=1}^\infty A_n := \left\{ \sum_{n=1}^\infty a_n : a_n \in A_n, n \geq 1 \right\},$$

and for a real number  $x$  and a set  $A$  let  $xA := \{xa : a \in A\} = Ax$ . Then by (3) the generalized Cantor set  $\Gamma_{\beta, \mathcal{D}'}$  of type  $\mathcal{D}'$  can be rewritten as

$$\Gamma_{\beta, \mathcal{D}'} = \sum_{n=1}^\infty D_n \beta^n.$$

Now we complete the proof of Theorem 2.2 by showing (II)  $\Rightarrow$  (III), which is based on Lemma 4.2 and Lemma 4.3. The difficulty in the proof is that a point in  $\Gamma_{\beta, \Omega_{2N-1}^\infty}$  may have multiple  $\Omega_{2N-1}^\infty$ -codes for  $\beta > 1/(2N-1)$ .

*Proof of Theorem 2.2* (II)  $\Rightarrow$  (III). Since  $N \geq 3$ , we have by assumption that  $0 < \beta \leq 1/[(3N-1)/2] < 1/N$ . By Lemma 4.2 there exists  $p \in \mathbb{Z}^+$  such that

$$(13) \quad \mathcal{D}' = \left( \bigotimes_{\ell=1}^p D'_\ell \right) \otimes \left( \bigotimes_{\ell=1}^p D'_{p+\ell} \right)^\infty \quad \text{with} \quad D'_\ell \subseteq D'_{p+\ell} \quad \text{for } 1 \leq \ell \leq p.$$

Moreover, we can require by Lemma 4.2 that  $\Gamma_{\beta, \mathcal{D}'}$  is generated by an IFS  $\{g_j(x) = \beta^p x + b_j : j \in J\}$ . Recall that  $0 \in \Gamma_{\beta, \mathcal{D}'}$ . Then we have  $b_j = g_j(0) \in \Gamma_{\beta, \mathcal{D}'}$  for all  $j \in J$ . Since  $0 < \beta < 1/N$ , any point in  $\Gamma_{\beta, \mathcal{D}'}$  has a unique  $\mathcal{D}'$ -code. Let  $\{b_{j,n}\}_{n=1}^\infty$  be the unique  $\mathcal{D}'$ -code of  $b_j$ , i.e.,

$$b_j = \pi(\{b_{j,n}\}_{n=1}^\infty) = \sum_{n=1}^\infty b_{j,n} \beta^n \quad \text{with } b_{j,n} \in D'_n \quad \text{for all } n \geq 1.$$

For  $1 \leq \ell \leq p$ , let  $B_\ell := \{b \in \mathbb{Z} : b + D'_\ell \subseteq D'_{p+\ell}\}$ . By Equation (13) the proof of (II)  $\Rightarrow$  (III) will then follow if we can show that  $D'_{p+\ell} = B_\ell + D'_\ell$  for  $1 \leq \ell \leq p$ . Clearly,  $0 \in B_\ell$  since  $D'_\ell \subseteq D'_{p+\ell}$ . Directly from the definition of sum of sets, we have the inclusion

$$B_\ell + D'_\ell = \bigcup_{b \in B_\ell} (b + D'_\ell) \subseteq D'_{p+\ell}.$$

On the other hand, let  $d \in D'_{p+\ell}$  for some  $1 \leq \ell \leq p$ . We split the proof of  $d \in B_\ell + D'_\ell$  into the following two cases.

Case I.  $d \leq [(N-1)/2]$ . Take

$$x = d\beta^{p+\ell} + \sum_{n=1}^{\infty} \lambda_{p+\ell+n} \beta^{p+\ell+n} \in \Gamma_{\beta, \mathcal{D}'},$$

where  $\lambda_n := \max\{d : d \in D'_n\}$  for  $n \geq 1$ . Since  $\Gamma_{\beta, \mathcal{D}'} = \bigcup_{j \in J} g_j(\Gamma_{\beta, \mathcal{D}'})$ , there exist  $j_x \in J$  and  $x' = \sum_{n=1}^{\infty} x'_n \beta^n = \pi(\{x'_n\}_{n=1}^{\infty}) \in \Gamma_{\beta, \mathcal{D}'}$  such that  $x = g_{j_x}(x')$ , i.e.,

$$(14) \quad \begin{aligned} d\beta^{p+\ell} + \sum_{n=1}^{\infty} \lambda_{p+\ell+n} \beta^{p+\ell+n} &= \sum_{n=1}^p b_{j_x, n} \beta^n + \sum_{n=1}^{\ell-1} (b_{j_x, p+n} + x'_n) \beta^{p+n} \\ &\quad + (b_{j_x, p+\ell} + x'_\ell) \beta^{p+\ell} + \sum_{n=1}^{\infty} (b_{j_x, p+\ell+n} + x'_{\ell+n}) \beta^{p+\ell+n}. \end{aligned}$$

Since  $0 < \beta < 1/N$ , we have  $x < \beta^{p+\ell-1}$ , and then we obtain that  $b_{j_x, n} = 0$  for  $1 \leq n < p+\ell$  and  $x'_n = 0$  for  $1 \leq n < \ell$ . Then Equation (14) can be rearranged in the following way:

$$(15) \quad d = (b_{j_x, p+\ell} + x'_\ell) + \sum_{n=1}^{\infty} (b_{j_x, p+\ell+n} + x'_{\ell+n} - \lambda_{p+\ell+n}) \beta^n.$$

Since  $0 < \beta < 1/N$  and the digits  $b_{j_x, n}$  satisfy  $b_{j_x, n} \leq \lambda_n$  for all  $n \geq 1$ , we have

$$\begin{aligned} \left| \sum_{n=1}^{\infty} (b_{j_x, p+\ell+n} + x'_{\ell+n} - \lambda_{p+\ell+n}) \beta^n \right| &\leq \sum_{n=1}^{\infty} |x'_{\ell+n} - (\lambda_{p+\ell+n} - b_{j_x, p+\ell+n})| \beta^n \\ &\leq \sum_{n=1}^{\infty} (N-1) \beta^n < 1. \end{aligned}$$

Then it follows from Equation (15) that

$$(16) \quad d = b_{j_x, p+\ell} + x'_\ell \in b_{j_x, p+\ell} + D'_\ell.$$

Since  $d \leq [(N-1)/2]$ , we have by Equation (16) that  $b_{j_x, p+\ell} \leq [(N-1)/2]$ . This, together with  $\lambda_\ell \leq N-1$  and  $0 < \beta \leq 1/[(3N-1)/2]$ , yields

$$b_{j_x, p+\ell} + \lambda_\ell \leq [(3N-1)/2] - 1 \leq [1/\beta] - 1,$$

i.e.,  $b_{j_x, p+\ell} + D'_\ell \subseteq \Omega_{[1/\beta]}$ . Note that

$$(17) \quad g_{j_x}(D'_\ell \beta^\ell) = \sum_{n \neq p+\ell} b_{j_x, n} \beta^n + (b_{j_x, p+\ell} + D'_\ell) \beta^{p+\ell} \subseteq \Gamma_{\beta, \mathcal{D}'} = \sum_{n=1}^{\infty} D'_n \beta^n.$$

Since  $0 < \beta \leq 1/[(3N-1)/2]$  and  $N \geq 3$ , we have  $b_{j_x, n} \leq N-1 < [1/\beta] - 1$  for  $n \geq 1$ . Then, by using Lemma 4.3 in Equation (17), we obtain  $b_{j_x, p+\ell} + D'_\ell \subseteq D'_{p+\ell}$ , i.e.,  $b_{j_x, p+\ell} \in B_\ell$ . So, by Equation (16) we have  $d \in B_\ell + D'_\ell$ .

Case II.  $d > [(N-1)/2]$ . Take

$$y = \lambda_{p+\ell-1} \beta^{p+\ell-1} + d\beta^{p+\ell} + \sum_{n=1}^{\infty} \lambda_{p+\ell+n} \beta^{p+\ell+n} \in \Gamma_{\beta, \mathcal{D}'},$$

Since  $\Gamma_{\beta, \mathcal{D}'} = \bigcup_{j \in J} g_j(\Gamma_{\beta, \mathcal{D}'})$ , there exist  $j_y \in J$  and  $y' = \sum_{n=1}^{\infty} y'_n \beta^n = \pi(\{y'_n\}_{n=1}^{\infty}) \in \Gamma_{\beta, \mathcal{D}'}$  such that  $y = g_{j_y}(y')$ , i.e.,

$$\begin{aligned}
 & \lambda_{p+\ell-1} \beta^{p+\ell-1} + d \beta^{p+\ell} + \sum_{n=1}^{\infty} \lambda_{p+\ell+n} \beta^{p+\ell+n} \\
 &= \sum_{n=1}^p b_{j_y, n} \beta^n + \sum_{n=1}^{\ell-2} (b_{j_y, p+n} + y'_n) \beta^{p+n} + (b_{j_y, p+\ell-1} + y'_{\ell-1}) \beta^{p+\ell-1} \\
 (18) \quad & + (b_{j_y, p+\ell} + y'_{\ell}) \beta^{p+\ell} + \sum_{n=1}^{\infty} (b_{j_y, p+\ell+n} + y'_{\ell+n}) \beta^{p+\ell+n}.
 \end{aligned}$$

By using  $0 < \beta < 1/N$  we have  $y < \beta^{p+\ell-2}$ , and then we obtain that

$$(19) \quad b_{j_y, n} = 0 \quad \text{for } 1 \leq n < p + \ell - 1$$

and  $y'_n = 0$  for  $1 \leq n < \ell - 1$ . Then Equation (18) can be rearranged as

$$\begin{aligned}
 \lambda_{p+\ell-1} &= (b_{j_y, p+\ell-1} + y'_{\ell-1}) + (b_{j_y, p+\ell} + y'_{\ell} - d) \beta \\
 (20) \quad & + \sum_{n=1}^{\infty} (b_{j_y, p+\ell+n} + y'_{\ell+n} - \lambda_{p+\ell+n}) \beta^{n+1},
 \end{aligned}$$

where we set  $y'_0 = 0$ . Since  $0 < \beta \leq 1/[(3N-1)/2]$  and  $d > [(N-1)/2]$ , we have

$$(21) \quad -([1/\beta] - 1) \leq b_{j_y, p+\ell} + y'_{\ell} - d \leq [(3N-1)/2] - 1 \leq [1/\beta] - 1.$$

In a similar way as in Case I, we can show that

$$(22) \quad \left| \sum_{n=1}^{\infty} (b_{j_y, p+\ell+n} + y'_{\ell+n} - \lambda_{p+\ell+n}) \beta^{n+1} \right| < \beta.$$

Then, by using Equation (21) and (22) it follows that

$$\left| (b_{j_y, p+\ell} + y'_{\ell} - d) \beta + \sum_{n=1}^{\infty} (b_{j_y, p+\ell+n} + y'_{\ell+n} - \lambda_{p+\ell+n}) \beta^{n+1} \right| < 1.$$

Then, by Equation (20) it follows that

$$(23) \quad \lambda_{p+\ell-1} = b_{j_y, p+\ell-1} + y'_{\ell-1}.$$

Substituting (23) in Equation (20) we obtain

$$d = (b_{j_y, p+\ell} + y'_{\ell}) + \sum_{n=1}^{\infty} (b_{j_y, p+\ell+n} + y'_{\ell+n} - \lambda_{p+\ell+n}) \beta^n.$$

This, again by using Equation (22), yields that

$$(24) \quad d = b_{j_y, p+\ell} + y'_{\ell} \in b_{j_y, p+\ell} + D'_{\ell}.$$

If  $b_{j_y, p+\ell} + \lambda_{\ell} < [1/\beta]$ , then  $b_{j_y, p+\ell} + D'_{\ell} \subseteq \Omega_{[1/\beta]}$ . In a similar way as in Equation (17), we can show that  $b_{j_y, p+\ell} + D'_{\ell} \subseteq D'_{p+\ell}$ , i.e.,  $b_{j_y, p+\ell} \in B_{\ell}$ . Thus, by Equation (24) we have  $d \in B_{\ell} + D'_{\ell}$ .

We will finish the proof of Case II by showing that  $b_{j_y, p+\ell} + \lambda_\ell \geq [1/\beta]$  will lead to a contradiction. Take  $z = y'_{\ell-1}\beta^{\ell-1} + \lambda_\ell\beta^\ell \in \Gamma_{\beta, \mathcal{D}'}$ . Then, by Equation (19) and (23), we have

$$\begin{aligned} g_{j_y}(z) &= (b_{j_y, p+\ell-1} + y'_{\ell-1})\beta^{p+\ell-1} + (b_{j_y, p+\ell} + \lambda_\ell)\beta^{p+\ell} + \sum_{n>p+\ell} b_{j_y, n}\beta^n \\ &= \lambda_{p+\ell-1}\beta^{p+\ell-1} + (b_{j_y, p+\ell} + \lambda_\ell)\beta^{p+\ell} + \sum_{n>p+\ell} b_{j_y, n}\beta^n. \end{aligned}$$

Since  $0 < \beta < 1/N$  and  $b_{j_y, p+\ell} + \lambda_\ell \geq [1/\beta]$ , we have

$$\begin{aligned} g_{j_y}(z) &> \lambda_{p+\ell-1}\beta^{p+\ell-1} + \sum_{n=p+\ell}^{\infty} (N-1)\beta^n \\ g_{j_y}(z) &< (\lambda_{p+\ell-1} + 1)\beta^{p+\ell-1} + \sum_{n=p+\ell}^{\infty} (N-1)\beta^n. \end{aligned}$$

Since  $g_{j_y}(z) \in \Gamma_{\beta, \mathcal{D}'} \subseteq \Gamma_{\beta, \Omega_N^\infty}$ , this implies that there exist  $c_n \in D'_n, n \geq p+\ell$  such that

$$(25) \quad g_{j_y}(z) = (\lambda_{p+\ell-1} + 1)\beta^{p+\ell-1} + \sum_{n=p+\ell}^{\infty} c_n\beta^n \in \Gamma_{\beta, \mathcal{D}'}.$$

Note that  $\lambda_{p+\ell-1} + 1 \leq N \leq [1/\beta] - 1$ , since from  $N \geq 3$  we have that  $0 < \beta < 1/[(3N-1)/2] \leq 1/(N+1)$ . Then, by using Lemma 4.3 in Equation (25), we obtain  $\lambda_{p+\ell-1} + 1 \in D'_{p+\ell-1}$ , leading to a contradiction with the definition of  $\lambda_{p+\ell-1}$ .  $\square$

## 5. PROOF OF THEOREM 2.3

In this section we will prove Theorem 2.3. It follows from Theorem 2.2 that (A)  $\Leftrightarrow$  (B)  $\Leftrightarrow$  (D). Clearly, (B)  $\Rightarrow$  (C). So it suffices to show (C)  $\Rightarrow$  (D).

Let  $\Gamma_{\beta, \mathcal{D}'}$  be a self-similar set generated by an IFS  $\{h_k(x) = \beta^{q_k}x + c_k : k \in K\}$  for some  $q_k \in \mathbb{Z}^+$ . Since  $0 \in \Gamma_{\beta, \mathcal{D}'}$ , there exists  $k \in K$  such that  $h_k(x) = \beta^{q_k}x$ . In a similar way as in the proof of Lemma 4.2, we can show that there exists  $p \in \mathbb{Z}^+$  such that

$$(26) \quad \{D'_n\}_{n=1}^\infty = \{D'_\ell\}_{\ell=1}^p \overline{\{D'_{p+\ell}\}_{\ell=1}^p} \quad \text{with } D'_\ell \subseteq D'_{p+\ell} \text{ for } 1 \leq \ell \leq p.$$

Let  $p_*$  be the least period of the periodic sequence of sets  $\overline{\{D'_{p+\ell}\}_{\ell=1}^p}$ . Clearly,  $p_*|p$ . We will show in the following lemma that  $p_*|q_k$  for all  $k \in K$ .

**Lemma 5.1.** *Let  $\Gamma_{\beta, \mathcal{D}'}$  be a generalized Cantor set of type  $\mathcal{D}' = \bigotimes_{n=1}^\infty D'_n$  as in (6) with span  $N = N_{\mathcal{D}'} \geq 2$ . Suppose  $0 < \beta \leq 1/(2N-1)$ . If  $\Gamma_{\beta, \mathcal{D}'}$  is a self-similar set generated by an IFS  $\{h_k(x) = r_k x + c_k : k \in K\}$  with  $r_k = \beta^{q_k}$  for some  $q_k \in \mathbb{Z}^+$ , then for any  $k \in K$  we have  $p_*|q_k$ . Moreover,  $c_k = \sum_{n=1}^{p+q_k} c_{k,n}\beta^n$ , where  $p$  is a period of  $\{D'_n\}_{n=1}^\infty$  as in (26).*

*Proof.* Since  $0 \in \Gamma_{\beta, \mathcal{D}'}$ , we have  $c_k = h_k(0) \in \Gamma_{\beta, \mathcal{D}'}$  for  $k \in K$ . Let  $\{c_{k,n}\}_{n=1}^\infty$  be the unique  $\mathcal{D}'$ -code of  $c_k$ . Here the uniqueness follows by using  $0 < \beta < 1/N$  in Lemma 4.3. Fix

$k \in K$ . Note that for  $n \geq 1$ ,

$$h_k(D'_n \beta^n) = (D'_n + c_{k,q_k+n})\beta^{q_k+n} + \sum_{\ell \neq q_k+n} c_{k,\ell} \beta^\ell \subseteq \sum_{\ell=1}^{\infty} D'_\ell \beta^\ell.$$

By using Lemma 4.3, this, together with  $0 < \beta \leq 1/(2N-1)$ , implies

$$(27) \quad D'_n + c_{k,q_k+n} \subseteq D'_{q_k+n} \quad \text{for all } n \geq 1.$$

Let  $p$  be an integer satisfying (26). Then

$$(28) \quad D'_{mp+n} = D'_{p+n} \quad \text{for all } m \geq 1 \text{ and } n \geq 1.$$

Using Equation (27) and (28), we obtain that, for  $n \geq 1$ ,

$$\begin{aligned} D'_{p+n} + \sum_{m=1}^p c_{k,p+mq_k+n} &= (D'_{p+n} + c_{k,p+q_k+n}) + \sum_{m=2}^p c_{k,p+mq_k+n} \\ &\subseteq D'_{p+q_k+n} + \sum_{m=2}^p c_{k,p+mq_k+n} \\ &= (D'_{p+q_k+n} + c_{k,p+2q_k+n}) + \sum_{m=3}^p c_{k,p+mq_k+n} \\ &\subseteq D'_{p+2q_k+n} + \sum_{m=3}^p c_{k,p+mq_k+n} \\ &\quad \dots \\ &\subseteq D'_{p+pq_k+n} = D'_{p+n}. \end{aligned}$$

This implies  $c_{k,p+mq_k+n} = 0$  for all  $1 \leq m \leq p$  and  $n \geq 1$ . In particular,  $c_{k,p+q_k+n} = 0$  for all  $n \geq 1$ , i.e.,  $c_k = \sum_{n=1}^{p+q_k} c_{k,n} \beta^n$ .

By (26) we have  $D'_n \subseteq D'_{p+n}$  for  $n \geq 1$ . Then,

$$h_k(D'_n \beta^{p+n}) = D'_n \beta^{p+q_k+n} + \sum_{n=1}^{p+q_k} c_{k,n} \beta^n \subseteq \Gamma_{\beta, \mathcal{D}'} = \sum_{\ell=1}^{\infty} D'_\ell \beta^\ell.$$

By using  $0 < \beta < 1/N$  in Lemma 4.3, this implies

$$(29) \quad D'_n \subseteq D'_{p+q_k+n} \quad \text{for all } n \geq 1.$$

By iteration of (29) and using (28), we obtain

$$D'_{p+n} \subseteq D'_{2p+q_k+n} = D'_{p+q_k+n} \subseteq D'_{2(p+q_k)+n} \subseteq \dots \subseteq D'_{p(p+q_k)+n} = D'_{p+n},$$

which yields

$$D'_{p+n} = D'_{p+q_k+n} \quad \text{for all } n \geq 1,$$

i.e.,  $q_k$  is a period of  $\overline{\{D'_{p+\ell}\}_{\ell=1}^p}$ . Thus, by the definition of  $p_*$  we have  $p_* | q_k$ .  $\square$

*Proof of Theorem 2.3 (C)  $\Rightarrow$  (D).* Let  $p$  be an integer satisfying (26), i.e.,

$$(30) \quad \mathcal{D}' = \left( \bigotimes_{\ell=1}^p D'_\ell \right) \otimes \left( \bigotimes_{\ell=1}^p D'_{p+\ell} \right)^\infty \quad \text{with } D'_\ell \subseteq D'_{p+\ell} \text{ for } 1 \leq \ell \leq p.$$

Suppose  $\Gamma_{\beta, \mathcal{D}'}$  is a self-similar set generated by an IFS  $\{h_k(x) = \beta^{q_k}x + b_k : k \in K\}$  with  $q_k \in \mathbb{Z}^+$ . We can require  $q_k > p$  for all  $k \in K$ , since otherwise we can replace the IFS  $\{h_k(x) = \beta^{q_k}x + c_k : k \in K\}$  by its finite iterations  $\{h_{k_1} \circ h_{k_2} \circ \dots \circ h_{k_m} : k_i \in K, i = 1, \dots, m\}$  for some large  $m$ . Let  $p_*$  be the least period of the sequence of sets  $\{\overline{D'_{p+\ell}}_{\ell=1}^p\}$ . Then  $p_*|p$ , and by Lemma 5.1 we have  $p_*|q_k$  for all  $k \in K$ . So,  $p_*(q_k - p)$  and then

$$(31) \quad D'_{q_k+n} = D'_{p+(q_k-p)+n} = D'_{p+n} \quad \text{for all } n \geq 1.$$

Set  $B_\ell := \{b \in \mathbb{Z} : b + D'_\ell \subseteq D'_{p+\ell}\}$  for  $1 \leq \ell \leq p$ . By Equation (30), we will finish the proof of (C)  $\Rightarrow$  (D) by showing  $D'_{p+\ell} = B_\ell + D'_\ell$  for all  $1 \leq \ell \leq p$ . Clearly,  $0 \in B_\ell$  by (30). By the definition of sum of sets we have

$$B_\ell + D'_\ell = \bigcup_{b \in B_\ell} (b + D'_\ell) \subseteq D'_{p+\ell}.$$

On the other hand, let  $d \in D'_{p+\ell}$  for some  $1 \leq \ell \leq p$ . Since  $p_*$  is the period of  $\{\overline{D'_{p+\ell}}_{\ell=1}^p\}$ , we have  $d \in D'_{p+\ell} = D'_{p+\ell+mp_*}$  for all  $m \geq 0$ . Take

$$z = \sum_{m=0}^{\infty} d\beta^{p+\ell+mp_*} \in \Gamma_{\beta, \mathcal{D}'}$$

Since  $\Gamma_{\beta, \mathcal{D}'} = \bigcup_{k \in K} h_k(\Gamma_{\beta, \mathcal{D}'})$ , there exists  $k_z \in K$  such that  $z \in h_{k_z}(\Gamma_{\beta, \mathcal{D}'})$ , i.e., (note by Lemma 5.1 that  $c_{k_z}$  has a finite expansion)

$$\begin{aligned} \sum_{m=0}^{\infty} d\beta^{p+\ell+mp_*} &= d\beta^{p+\ell} + d\beta^{p+p_*} + \dots + d\beta^{q_{k_z}+\ell} + \dots \\ &\in \sum_{n=1}^{q_{k_z}} c_{k_z, n} \beta^n + \sum_{n=1}^p (c_{k_z, q_{k_z}+n} + D'_n) \beta^{q_{k_z}+n} + \sum_{n=1}^{\infty} D'_{p+n} \beta^{q_{k_z}+p+n}, \end{aligned}$$

where the equality holds since  $q_{k_z} > p$  and  $p_*(q_{k_z} - p)$ . By using  $0 < \beta \leq 1/(2N-1)$  in Lemma 4.3, this implies

$$(32) \quad d \in c_{k_z, q_{k_z}+\ell} + D'_\ell.$$

Again by using Lemma 4.3 in the following equation

$$h_{k_z}(D'_\ell \beta^\ell) = (c_{k_z, q_{k_z}+\ell} + D'_\ell) \beta^{q_{k_z}+\ell} + \sum_{n \neq q_{k_z}+\ell} c_{k_z, n} \beta^n \subseteq \sum_{n=1}^{\infty} D'_n \beta^n,$$

and using Equation (31), we have

$$c_{k_z, q_{k_z}+\ell} + D'_\ell \subseteq D'_{q_{k_z}+\ell} = D'_{p+\ell},$$

i.e.,  $c_{k_z, q_{k_z}+\ell} \in B_\ell$ . Thus, by Equation (32) we obtain  $d \in B_\ell + D'_\ell$ .  $\square$

At the end of this section, we make some remarks on Theorem 2.3.

**Remark 5.1.** When the sequence of digit sets  $\{D_n\}_{n=1}^\infty$  is eventually uniform, i.e., there exist a large number  $M > 0$  and a nonnegative integer  $\tau$  such that  $D_n = \{\gamma_n, \gamma_n + \tau, \dots, \gamma_n + m_n \tau\}$  with  $m_n \in \mathbb{Z}^+$  for all  $n \geq M$ , we can add the following condition into the equivalent condition list in Theorem 2.3.

(E)  $\Gamma_{\beta, \mathcal{D}}$  is a self-similar set generated by an IFS  $\{S_u(x) = r_u x + e_u : u \in U\}$  with  $0 < r_u < 1$  for all  $u \in U$ .

It suffices to show (E)  $\Rightarrow$  (C). Suppose  $\Gamma_{\beta, \mathcal{D}'}$  is a self-similar set generated by an IFS  $\{S_u(x) = r_u x + e_u : u \in U\}$  with  $r_u > 0$  for all  $u \in U$ . In a similar way as in the proof of Lemma 4.1 and Lemma 4.2, we can show that there exists  $p \in \mathbb{Z}^+$  such that

$$(33) \quad \{D'_n\}_{n=1}^\infty = \{D'_\ell\}_{\ell=1}^p \overline{\{D'_{p+\ell}\}_{\ell=1}^p} \quad \text{with } D'_\ell \subseteq D'_{p+\ell} \text{ for } 1 \leq \ell \leq p.$$

Under the eventually uniform condition, all the digit sets  $D'_n, n \geq 1$ , can be written as  $\tau$  times a consecutive digit set, i.e.,

$$(34) \quad D'_n = D_n - \gamma_n = \{0, \tau, \dots, m_n \tau\} = \tau \{0, 1, \dots, m_n\}.$$

Then, we have by Equation (33) and (34) that

$$\bigcup_{n=1}^\infty D'_n \subseteq \{0, \tau, 2\tau, \dots, m^* \tau\},$$

where  $m^* = \max_{1 \leq \ell \leq 2p} m_\ell$ . In a similar way as in the proof of Lemma 4.1—Lemma 4.4 in [LYZ11], we can show  $r_u = \beta^{q_u}$  with  $q_u \in \mathbb{Z}^+$  for  $u \in U$ .

**Remark 5.2.** When the sequence of digit sets  $\{D_n\}_{n=1}^\infty$  is eventually uniform and  $\Gamma_{\beta, \mathcal{D}}$  is symmetric, we can add another equivalent condition which allows us to replace the restriction  $0 < r_u < 1$  in (E) by  $0 < |r_u| < 1$ .

## 6. INTERSECTIONS OF GENERALIZED CANTOR SETS

Let  $\Gamma_{\beta, \mathcal{C}}$  and  $\Gamma_{\beta, \mathcal{D}}$  be two generalized Cantor sets of types  $\mathcal{C} = \bigotimes_{n=1}^\infty C_n$  and  $\mathcal{D} = \bigotimes_{n=1}^\infty D_n$ , respectively. Using (3), one can easily write the intersection  $\Gamma_{\beta, \mathcal{C}} \cap \Gamma_{\beta, \mathcal{D}}$  as

$$\Gamma_{\beta, \mathcal{C}} \cap \Gamma_{\beta, \mathcal{D}} = \left\{ \sum_{n=1}^\infty d_n \beta^n : d_n \in C_n \cap D_n \right\} = \pi(\mathcal{C} \cap \mathcal{D}),$$

where  $\mathcal{C} \cap \mathcal{D} := \bigotimes_{n=1}^\infty (C_n \cap D_n)$ . So  $\Gamma_{\beta, \mathcal{C}} \cap \Gamma_{\beta, \mathcal{D}}$  is also a generalized Cantor set if  $\Gamma_{\beta, \mathcal{C}} \cap \Gamma_{\beta, \mathcal{D}} \neq \emptyset$ . Using Theorem 2.2, we have the following characterization for the self-similar structure of intersections of generalized Cantor sets.

**Theorem 6.1.** Let  $\Gamma_{\beta, \mathcal{C}}$  and  $\Gamma_{\beta, \mathcal{D}}$  be two generalized Cantor sets of types  $\mathcal{C} = \bigotimes_{n=1}^\infty C_n$  and  $\mathcal{D} = \bigotimes_{n=1}^\infty D_n$  respectively. Let  $N = N_{\mathcal{C} \cap \mathcal{D}} \geq 3$  be the span of  $\mathcal{C} \cap \mathcal{D} = \bigotimes_{n=1}^\infty (C_n \cap D_n)$ . Suppose  $0 < \beta \leq 1/(3N - 1)/2$ . Then the following conditions are equivalent.

- (I)  $\Gamma_{\beta, \mathcal{C}} \cap \Gamma_{\beta, \mathcal{D}}$  is self-similar set generated by an IFS  $\{f_i(x) = rx + a_i : i \in I\}$ ;
- (II)  $\Gamma_{\beta, \mathcal{C}} \cap \Gamma_{\beta, \mathcal{D}}$  is self-similar set generated by an IFS  $\{g_j(x) = rx + b_j : j \in J\}$  with  $r = \beta^q$  for some  $q \in \mathbb{Z}^+$ ;

(III) the sequence of sets  $\{C_n \cap D_n - \gamma_n\}_{n=1}^\infty$  is SEP, where  $\gamma_n = \min\{d : d \in C_n \cap D_n\}$ .

Similarly, by using Theorem 2.3 we have more about the self-similar structure of the intersections  $\Gamma_{\beta, \mathcal{C}} \cap \Gamma_{\beta, \mathcal{D}}$  for  $0 < \beta \leq 1/(2N_{\mathcal{C} \cap \mathcal{D}} - 1)$ . Moreover, by using Theorem 2.1 we generalize results in [LYZ11, KLD10] on the self-similar structure of intersections if all the digit sets  $C_n \cap D_n, n \geq 1$ , are consecutive.

In particular, we consider the self-similar structure of intersections of a generalized Cantor set  $\Gamma_{\beta, \mathcal{D}}$  with its translations, i.e.,  $\Gamma_{\beta, \mathcal{D}} \cap (\Gamma_{\beta, \mathcal{D}} + t)$  for  $t \in \mathbb{R}$ . Clearly,

$$\Gamma_{\beta, \mathcal{D}} \cap (\Gamma_{\beta, \mathcal{D}} + t) \neq \emptyset \quad \text{if and only if} \quad t \in \Gamma_{\beta, \mathcal{D}} - \Gamma_{\beta, \mathcal{D}}.$$

Using Equation (3) the difference set  $\Gamma_{\beta, \mathcal{D}} - \Gamma_{\beta, \mathcal{D}}$  can be written as

$$\Gamma_{\beta, \mathcal{D}} - \Gamma_{\beta, \mathcal{D}} = \left\{ \sum_{n=1}^{\infty} t_n \beta^n : t_n \in D_n - D_n \right\} = \pi(\mathcal{D} - \mathcal{D}),$$

where  $\mathcal{D} - \mathcal{D} := \bigotimes_{n=1}^{\infty} (D_n - D_n)$ . Then, for  $t \in \Gamma_{\beta, \mathcal{D}} - \Gamma_{\beta, \mathcal{D}}$  we can show in a similar way as in [LYZ11] that

$$\Gamma_{\beta, \mathcal{D}} \cap (\Gamma_{\beta, \mathcal{D}} + t) = \bigcup_{\{t_n\}_{n=1}^{\infty}} \pi \left( \bigotimes_{n=1}^{\infty} (D_n \cap (D_n + t_n)) \right),$$

where the union is taken over all  $\mathcal{D} - \mathcal{D}$ -codes  $\{t_n\}_{n=1}^{\infty}$  of  $t$ . If  $t \in \Gamma_{\beta, \mathcal{D}} - \Gamma_{\beta, \mathcal{D}}$  has a unique  $\mathcal{D} - \mathcal{D}$ -code  $\{t_n\}_{n=1}^{\infty}$ , then

$$\Gamma_{\beta, \mathcal{D}} \cap (\Gamma_{\beta, \mathcal{D}} + t) = \pi \left( \bigotimes_{n=1}^{\infty} (D_n \cap (D_n + t_n)) \right),$$

which is a generalized Cantor set of type  $\bigotimes_{n=1}^{\infty} (D_n \cap (D_n + t_n))$ .

Let  $\mathcal{U}_{\beta, \mathcal{D} - \mathcal{D}}$  be the set of  $t \in \Gamma_{\beta, \mathcal{D}} - \Gamma_{\beta, \mathcal{D}}$  which has a unique  $\mathcal{D} - \mathcal{D}$ -code, i.e.,

$$\mathcal{U}_{\beta, \mathcal{D} - \mathcal{D}} := \{t \in \Gamma_{\beta, \mathcal{D}} - \Gamma_{\beta, \mathcal{D}} : |\pi^{-1}(t)| = 1\}.$$

When  $0 < \beta < 1/(2N_{\mathcal{D}} - 1)$ , the projection map  $\pi : \mathcal{D} - \mathcal{D} \rightarrow \Gamma_{\beta, \mathcal{D}} - \Gamma_{\beta, \mathcal{D}}$  is bijective. So any point in  $\Gamma_{\beta, \mathcal{D}} - \Gamma_{\beta, \mathcal{D}}$  has a unique  $\mathcal{D} - \mathcal{D}$ -code, i.e.,  $\mathcal{U}_{\beta, \mathcal{D} - \mathcal{D}} = \Gamma_{\beta, \mathcal{D}} - \Gamma_{\beta, \mathcal{D}}$ . When  $1/(2N_{\mathcal{D}} - 1) \leq \beta \leq 1/[(3N_{\mathcal{D}} - 1)/2]$ , still there exist infinitely many  $t \in \Gamma_{\beta, \mathcal{D}} - \Gamma_{\beta, \mathcal{D}}$  having a unique  $\mathcal{D} - \mathcal{D}$ -code (if all the digit sets  $\{D_n\}_{n=1}^{\infty}$  are consecutive, see for example [KLD10]).

By using Theorem 2.2, we have the following theorem on the self-similar structure of intersections of a generalized Cantor set with its translations.

**Theorem 6.2.** *Let  $\Gamma_{\beta, \mathcal{D}}$  be a generalized Cantor set of type  $\mathcal{D} = \bigotimes_{n=1}^{\infty} D_n$  with span  $N_{\mathcal{D}} \geq 3$  in (3). Suppose  $0 < \beta \leq 1/[(3N_{\mathcal{D}} - 1)/2]$  and  $t \in \mathcal{U}_{\beta, \mathcal{D} - \mathcal{D}}$ . Then the following conditions are equivalent.*

- (I)  $\Gamma_{\beta, \mathcal{D}} \cap (\Gamma_{\beta, \mathcal{D}} + t)$  is self-similar set generated by an IFS  $\{f_i(x) = rx + a_i : i \in I\}$ ;
- (II)  $\Gamma_{\beta, \mathcal{D}} \cap (\Gamma_{\beta, \mathcal{D}} + t)$  is self-similar set generated by an IFS  $\{g_j(x) = rx + b_j : j \in J\}$  with  $r = \beta^q$  for some  $q \in \mathbb{Z}^+$ ;



(III) the sequence of sets  $\{D_n \cap (D_n + t_n) - \gamma_{t,n}\}_{n=1}^\infty$  is SEP, where  $\gamma_{t,n} = \min\{d : d \in D_n \cap (D_n + t_n)\}$ .

Similarly, when  $0 < \beta \leq 1/(2N-1)$  (or when  $D_n$  are consecutive for all  $n \geq 1$ ), we have more on the self-similar structure of intersections  $\Gamma_{\beta, \mathcal{D}} \cap (\Gamma_{\beta, \mathcal{D}} + t)$  by using Theorem 2.3 (or Theorem 2.1).

## 7. FINAL REMARKS AND OPEN QUESTIONS

We give an example to illustrate that the upper bound  $1/[(3N-1)/2]$  for  $\beta$  in Theorem 2.2 can not be improved to  $1/N$ .

**Example 7.1.** Let  $D_1 = \{0\}$ ,  $D_2 = \{0, 4\}$ ,  $D_{2m+1} = \{0, 1\}$ ,  $D_{2m+2} = \{0, 2, 4\}$  for all  $m \geq 1$ . Clearly, the span  $N$  of  $\mathcal{D} = \bigotimes_{n=1}^\infty D_n$  equals 5, and the sequence of sets  $\{D_n\}_{n=1}^\infty$  is not SEP. Take  $\beta = 1/6$ . Then  $1/[(3N-1)/2] < \beta < 1/N$ . We will show that  $\Gamma_{\beta, \mathcal{D}}$  is a self-similar set generated by an IFS  $\{g_j(x) = rx + b_j : j \in J\}$  with  $r = \beta^2$ .

Recall that  $\pi$  is the coding map from  $\mathcal{D}$  to  $\Gamma_{\beta, \mathcal{D}}$  defined by letting

$$\pi(\{d_n\}_{n=1}^\infty) = \sum_{n=1}^\infty d_n \beta^n \quad \text{for } \{d_n\}_{n=1}^\infty \in \mathcal{D}.$$

Let  $\{g_j(x) = \beta^2 x + b_j\}_{j=1}^8$  be the sequence of similitudes, where

$$\begin{aligned} b_1 &= \pi(0000\bar{0}), & b_2 &= \pi(0002\bar{0}), & b_3 &= \pi(0004\bar{0}), & b_4 &= \pi(0010\bar{0}); \\ b_5 &= \pi(0400\bar{0}), & b_6 &= \pi(0402\bar{0}), & b_7 &= \pi(0404\bar{0}), & b_8 &= \pi(0410\bar{0}). \end{aligned}$$

Then, by using  $D_n = D_{n+2}$  for  $k \geq 3$ , we have

$$\begin{aligned} g_1(\Gamma_{\beta, \mathcal{D}}) &= g_1\left(\sum_{n=1}^\infty D_n \beta^n\right) = D_1 \beta^3 + D_2 \beta^4 + \sum_{n=3}^\infty D_n \beta^{n+2} \\ &= D_1 \beta^3 + D_2 \beta^4 + \sum_{n=5}^\infty D_n \beta^n. \end{aligned}$$

Similarly,

$$\begin{aligned} g_2(\Gamma_{\beta, \mathcal{D}}) &= D_1 \beta^3 + (D_2 + 2) \beta^4 + \sum_{n=5}^\infty D_n \beta^n, \\ g_3(\Gamma_{\beta, \mathcal{D}}) &= D_1 \beta^3 + (D_2 + 4) \beta^4 + \sum_{n=5}^\infty D_n \beta^n, \\ g_4(\Gamma_{\beta, \mathcal{D}}) &= (D_1 + 1) \beta^3 + D_2 \beta^4 + \sum_{n=5}^\infty D_n \beta^n. \end{aligned}$$

Then, by using  $\beta = 1/6$  we obtain that

$$\bigcup_{j=1}^4 g_j(\Gamma_{\beta, \mathcal{D}}) = \sum_{n=3}^\infty D_n \beta^n.$$

In a similar way, one can also show that  $\bigcup_{j=5}^8 g_j(\Gamma_{\beta, \mathcal{D}}) = 4\beta^2 + \sum_{n=3}^{\infty} D_n \beta^n$ . Hence,

$$\bigcup_{j=1}^8 g_j(\Gamma_{\beta, \mathcal{D}}) = \sum_{n=1}^{\infty} D_n \beta^n = \Gamma_{\beta, \mathcal{D}}.$$

We end the paper with some open questions:

- Q1. Is the upper bound  $1/[3(N-1)/2]$  for  $\beta$  in Theorem 2.2 sharp? If not, what is the critical upper bound for  $\beta$ ?
- Q2. Can we add Condition (E) in the equivalent condition list in Theorem 2.3 without the eventually uniform assumption on the digit sets  $\{D_n\}_{n=1}^{\infty}$  as described in Remark 5.1.
- Q3. For  $\beta > 1/(2N-1)$ , by analogy with [KLD10, Theorem 4.6] there should be a critical point  $\beta_c(\mathcal{D}) (\geq 1/(2N-1))$  such that  $\mathcal{U}_{\beta, \mathcal{D}-\mathcal{D}}$  has positive Hausdorff dimension if  $0 < \beta < \beta_c(\mathcal{D})$ , and  $\mathcal{U}_{\beta, \mathcal{D}-\mathcal{D}}$  contains countably many points if  $\beta_c(\mathcal{D}) < \beta < 1/N_{\mathcal{D}}$ . How to characterize the set  $\mathcal{U}_{\beta, \mathcal{D}-\mathcal{D}}$  (cf. [Ped05])? What is the critical point  $\beta_c(\mathcal{D})$ ?
- Q4. Let  $\mathcal{S}_{\beta, \mathcal{D}-\mathcal{D}}$  be the set of  $t \in \mathcal{U}_{\beta, \mathcal{D}-\mathcal{D}}$  which makes the intersection  $\Gamma_{\beta, \mathcal{D}} \cap (\Gamma_{\beta, \mathcal{D}} + t)$  a self-similar set generated by an IFS  $\{f_i(x) = rx + a_i : i \in I\}$ . According to Theorem 6.2 and by analogy with [KLD10, Theorem 5.1], there should exist a critical point  $\alpha_c(\mathcal{D}) (\leq \beta_c(\mathcal{D}))$  such that  $\mathcal{S}_{\beta, \mathcal{D}-\mathcal{D}}$  has positive Hausdorff dimension if  $0 < \beta < \alpha_c(\mathcal{D})$ , and  $\mathcal{S}_{\beta, \mathcal{D}-\mathcal{D}}$  contains countably many points if  $\alpha_c(\mathcal{D}) < \beta < 1/[(3N_{\mathcal{D}}-1)/2]$ . What is the critical point  $\alpha_c(\mathcal{D})$ ?

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